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Uniform Inequalities for Ultraspherical Polynomials and Bessel Functions of Fractional Order

A. K. COMMON

Mathematical Institute, University of Kent, Kent CT2 7NZ, England

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In a recent paper, L. Lorch [1] has refined a standard inequality [2, Theorem 7.33.2, p. 171] for ultraspherical (Gegenbauer) polynomials of degree n and parameter α by proving that

$$|P_n^{(\alpha)}(\cos\theta)| < 2^{1-\alpha}(n+\alpha)^{\alpha-1}(\sin\theta)^{-\alpha}[\Gamma(\alpha)]^{-1} \equiv A_n(\cos\theta), \qquad 0 \leqslant \theta \leqslant \pi \tag{1}$$

for $n = 0, 1, 2, \ldots$; $0 < \alpha < 1$. This has been proved previously for the special case of Legendre polynomials, where $\alpha = \frac{1}{2}$, by V. A. Antonov and K. V. Holševnikov [3] and earlier still by A. Martin [4].

This bound has the disadvantage that it tends to infinity as $\theta \to 0$, π, and this has led the author of [4], in a study of the high energy behaviour of scattering amplitudes at fixed scattering angle, to construct the following "uniform" bound for Legendre polynomials $P_n(\cos\theta)$:

$$|P_n(\cos\theta)| \leqslant [1 + n(n+1)(1-\cos^2\theta)]^{-1/4}, \qquad n = 0, 1, 2, \ldots; 0 \leqslant \theta \leqslant \pi. \tag{2}$$

There are three nice features of this bound:

(i) It remains finite ($\leqslant 1$) for all $0 \leqslant \theta \leqslant \pi$.

(ii) For fixed θ in $[0, \pi]$, it has the correct behaviour (up to a constant factor $(\pi/2)^{1/2}$) as $n \to \infty$.

(iii) It is very good near $\theta = 0$ and π since the bound and the function being bounded have the same modulus at these values of θ and similarly for their first derivatives.

We will follow closely the methods of [4] to extend these results to the case of general ultraspherical polynomials by proving that for $n = 0, 1, 2, \ldots$ and all $0 \leqslant \theta \leqslant \pi$,

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$$|P_n^{(\alpha)}(\cos\theta)| \leqslant P_n^{(\alpha)}(1)\left[1+\frac{P_n^{(\alpha)}(1)'}{\beta(\alpha)\,P_n^{(\alpha)}(1)}(1-\cos^2\theta)\right]^{-\alpha/2}$$
$$\equiv B_n(\cos\theta), \tag{3}$$

where

$$\begin{aligned} \beta &= \alpha; & 1>\alpha\geqslant 0.065 \\ &= \mathrm{Max}[\alpha, f(\alpha)]; & 0.065>\alpha>0 \end{aligned} \tag{4}$$

with

$$f(\alpha)=\frac{1}{2(\alpha+1)}\left\{\left[\frac{\Gamma(\alpha+1)(1+2\alpha)(2+\alpha)^{1-2\alpha}}{2^{1-\alpha}}\right]^{2/\alpha}-0.503\right\}^{-1}. \tag{5}$$

The bound (3) again has the good properties (i)–(iii) except that when $0.065>\alpha>0$ the bound and the polynomial being bounded may not have the same derivative at $\theta=0, \pi$. Also our bounds reduce to the bound given in [4] for $\alpha=\frac{1}{2}$. They also lead in the limit $n\to\infty$ to the following bound on Bessel functions;

$$|J_{\alpha-\frac{1}{2}}(z)| \leqslant |z/2|^{\alpha-\frac{1}{2}}\,\Gamma(\alpha+\tfrac{1}{2})^{-1}\{1+z^2/[(2\alpha+1)\,\beta(\alpha)]\}^{-\alpha/2}, \quad 0<\alpha<1 \tag{6}$$

for all real z where $\beta(\alpha)$ is defined in (4).

1. Bounds Near $\theta=0$ and π

Since the ultraspherical polynomials are even or odd functions we need to consider only values of θ in the interval $[0, \pi/2]$ and prove the following result:

Theorem 1. *Let $\theta=\theta_1$ be the zero of $P_n^{(\alpha)}(\cos\theta)$ nearest $\theta=0$, then*

$$P_n^{(\alpha)}(\cos\theta) \leqslant P_n^{(\alpha)}(1)\left\{1+\frac{P_n^{(\alpha)}(1)'}{\alpha P_n^{(\alpha)}(1)}(1-\cos^2\theta)\right\}^{-\alpha/2}, \qquad 0\leqslant\theta\leqslant\theta_1 \tag{7}$$

for $n=0, 1, 2,\ldots, 0<\alpha<1$.

Proof. Let $y(x)=P_n^{(\alpha)}(\cos\theta)$, where $x=\cos\theta$. Then from the defining equation for ultraspherical polynomials,

$$\frac{d}{dx}\{(1-x^2)^{1+\alpha}y'(x)\}=(1-x^2)^{(1+\alpha)}y''(x)-2x(1+\alpha)y'(x)(1-x^2)^{\alpha}$$

$$=-(1-x^2)^{\alpha}n(n+2\alpha)y(x)-xy'(x)(1-x^2)^{\alpha}.$$

Therefore

$$(1-x^2)^{1+\alpha}y'(x)=\int_x^1[(1-t^2)^{\alpha}n(n+2\alpha)y(t)+ty'(t)(1-t^2)^{\alpha}]\,dt$$

$$\geqslant\int_x^1(1-t^2)^{\alpha}\,dt\,n(n+2\alpha)y(x)+\tfrac{1}{2}\int_x^1 2t(1-t^2)^{\alpha}\,dt\,y'(x)$$

$$\geqslant\{n(n+2\alpha)(1-x)^{\alpha+1}(1+x)^{\alpha}y(x)$$

$$+\tfrac{1}{2}(1-x^2)^{\alpha+1}y'(x)\}/(\alpha+1),$$

where use has been made of the convexity of $y(x)$ in the interval $[\cos\theta_1, 1]$. Therefore for these values of x,

$$y'(x)\geqslant\frac{2n(n+2\alpha)\,y(x)}{(1+x)(2\alpha+1)}=\frac{2}{(1+x)}y(x)\frac{y'(1)}{y(1)}. \tag{8}$$

The function,

$$f(x)=1-\frac{y^{2/\alpha}(x)}{y^{2/\alpha}(1)}\left[1-\frac{y'(1)}{\alpha y(1)}(1-x^2)\right]$$

has the properties that $f(1)=0$ and $f'(x)\leqslant 0$ for $\cos\theta_1\leqslant x\leqslant 1$, on using (8). Therefore $f(x)\geqslant 0$ and the theorem follows immediately.

2. Uniform Bounds

To extend the bound (7) to the whole interval $0\leqslant\theta\leqslant\pi$, we will attempt to show that it is *worse* than that given by (1) for all $\theta_1\leqslant\theta\leqslant\pi-\theta_1$. This we have found possible for $0.065<\alpha<1$ but for smaller values of α we have had to take a slightly modified bound. Note that for $n=0, 1$, the result given by (7) already covers the whole interval so for the remainder of this section we take $n\geqslant 2$. First we need the following result.

LEMMA 1. *For* $\frac{1}{2}\leqslant\alpha\leqslant 1$; $n=2, 3,\ldots$,

$$\left[\frac{\Gamma(\alpha)(n+\alpha)^{(1-\alpha)}P_n^{(\alpha)}(1)}{2^{1-\alpha}}\right]^{2/\alpha}-\frac{n(n+2\alpha)}{\alpha(2\alpha+1)}\geqslant\frac{n^2}{4}\left\{\frac{\pi^{1/\alpha}}{\Gamma(\alpha+\frac{1}{2})^{2/\alpha}}-\frac{4}{\alpha(2\alpha+1)}\right\}. \tag{9}$$

Proof. The left-hand side

$$= \left[\frac{\Gamma(n+2\alpha)\,\Gamma(\alpha)(n+\alpha)^{(1-\alpha)}}{\Gamma(n+1)\,2^{1-\alpha}\Gamma(2\alpha)}\right]^{2/\alpha} - \frac{n(n+2\alpha)}{\alpha(2\alpha+1)}$$

$$\geqslant \left[\frac{\Gamma(\alpha)(n+\alpha)^{(1-\alpha)}}{\Gamma(2\alpha)\,2^{1-\alpha}(n+2\alpha)^{1-2\alpha}}\right]^{2/\alpha} - \frac{n(n+2\alpha)}{\alpha(2\alpha+1)}$$

$$\geqslant (n+\alpha)^2 \left\{\left(\frac{n+\alpha}{n+2\alpha}\right)^{(2/\alpha-4)} \left[\frac{\Gamma(\alpha)}{\Gamma(2\alpha)\,2^{1-\alpha}}\right]^{2/\alpha} - \frac{1}{\alpha(2\alpha+1)}\right\}$$

$$\geqslant \frac{n^2}{4}\left\{\frac{\pi^{1/\alpha}}{[\Gamma(\alpha+\frac{1}{2})]^{2/\alpha}} - \frac{4}{2(2\alpha+1)}\right\},$$

where inequality (8) of [1] has been used to obtain the second line. There is another result we will have to use in the case when $0<\alpha<\frac{1}{2}$ and it is given by the following lemma.

LEMMA 2. *When* $0<\alpha<\frac{1}{2}$, $u_n=\Gamma(n+2\alpha)(n+\alpha)^{1-2\alpha}/\Gamma(n+1)$ *increases as n increases for* $n=2, 3, 4,\ldots$

Proof.

$$\frac{u_n}{u_{n+1}} = \left[1+\frac{(1-2\alpha)}{n+2\alpha}\right]\left[1-\frac{1}{n+\alpha+1}\right]^{1-2\alpha}$$

$$< \left[1+\frac{(1-2\alpha)}{n+2\alpha}\right]\left[1-\frac{(1-2\alpha)}{n+\alpha+1}-\frac{\alpha(1-2\alpha)}{(n+\alpha+1)^2}-\frac{\alpha(1-2\alpha)(1+2\alpha)}{3(n+\alpha+1)^3}\right]$$

$$\leqslant 1+\frac{(1-2\alpha)\,\alpha}{(n+2\alpha)(n+\alpha+1)^2}\left[\alpha-\left(\frac{1+2\alpha}{3}\right)\left(\frac{n+2\alpha}{n+\alpha+1}\right)\right].$$

The R.H.S. is less than one for $0<\alpha<\frac{1}{2}$; $n=2, 3,\ldots$ so u_n increases with n.

Two more lemmas are needed to extend our bound to the whole interval $0\leqslant\cos\theta\leqslant 1$.

LEMMA 3.

$$f(\alpha)\equiv\pi^{1/\alpha}/[\Gamma(\alpha+\tfrac{1}{2})]^{2/\alpha}$$

is a decreasing function of α *in the interval* $[\frac{1}{2}, 1]$.

Proof.

$$df(\alpha)/d\alpha = f(\alpha)\left\{-\frac{1}{\alpha^2}\,[\log\pi - 2\log\Gamma(\alpha+\tfrac{1}{2})] - \left(\frac{2}{\alpha}\right)\frac{\Gamma'(\alpha+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})}\right\}.$$

For $\frac{3}{4}\leqslant\alpha\leqslant 1$, $\log\Gamma(\alpha+\frac{1}{2})<0$ and $-(2/\alpha)(\Gamma'(\alpha+\frac{1}{2})/\Gamma(\alpha+\frac{1}{2}))\leqslant 0.5/\alpha$ so that,

$$\frac{df}{d\alpha}\leqslant -\frac{1.15}{\alpha^2}+\frac{0.5}{\alpha}\leqslant 0. \tag{10}$$

Similarly for $\frac{1}{2}\leqslant\alpha<\frac{3}{4}$,

$$\frac{df}{d\alpha}\leqslant -\frac{1.15}{\alpha^2}-\left(\frac{2}{\alpha}\right)\frac{\Gamma'(1)}{\Gamma(1)}\leqslant 0, \tag{11}$$

so the lemma is proved.

LEMMA 4. *For* $\frac{1}{2}>\alpha\geqslant 0.065$,

$$\alpha\geqslant\frac{1}{2(\alpha+1)}\left\{\left[\frac{\Gamma(\alpha+1)(1+2\alpha)(2+\alpha)^{1-2\alpha}}{2^{1-\alpha}}\right]^{2/\alpha}-0.503\right\}^{-1}. \tag{12}$$

Proof. First take the inequality in the form

$$g(\alpha)\equiv\Gamma(\alpha+1)(1+2\alpha)(2+\alpha)^{1-3\alpha/2}\,2^{\alpha-1}$$
$$\geqslant\left\{0.503(2+\alpha)\left(1+\frac{1}{\alpha(1+\alpha)}\right)\right\}^{\alpha/2}\equiv h(\alpha). \tag{13}$$

Now

$$\frac{d}{d\alpha}\left[1+\frac{1}{\alpha(1+\alpha)}\right]^{\alpha/2}$$
$$=\frac{1}{2}\left[1+\frac{1}{\alpha(1+\alpha)}\right]^{\alpha/2}\left\{\log\left[1+\frac{1}{\alpha(1+\alpha)}\right]-\frac{(1+2\alpha)}{[1+\alpha(\alpha+1)](1+\alpha)}\right\}. \tag{14}$$

For $0<\alpha<\frac{1}{2}$,

$$\frac{d}{d\alpha}\left\{\log\left[1+\frac{1}{\alpha(1+\alpha)}\right]-\frac{(1+2\alpha)}{[1+\alpha(\alpha+1)][1+\alpha]}\right\}\leqslant 0. \tag{15}$$

Therefore

$$\frac{d}{d\alpha}\left[1+\frac{1}{\alpha(1+\alpha)}\right]^{\alpha/2}\geqslant\frac{1}{2}\left[1+\frac{1}{\alpha(1+\alpha)}\right]^{\alpha/2}\left\{\log\frac{7}{3}-\frac{16}{21}\right\}\geqslant 0, \tag{16}$$

so that $h(\alpha)$ defined as the R.H.S. of (13) is an increasing function of α in $(0, \frac{1}{2})$. Similarly if $g(\alpha)$ is defined as the L.H.S. of (13), then

$$dg(\alpha)/d\alpha = \Gamma(\alpha+1)(1+2\alpha)(2+\alpha)^{(1-3\alpha/2)}2^{\alpha-1}\left\{\frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)}\right.$$
$$\left.+\frac{2}{1+2\alpha}-\frac{3}{2}\log(2+\alpha)+\frac{(1-3\alpha/2)}{(2+\alpha)}+\log_e 2\right\}. \quad (17)$$

By using the fact that $(\Gamma'(\alpha+1))/(\Gamma(\alpha+1))$ is a monotonic increasing function of α in the intervals $(0, \frac{1}{4})$ and $(\frac{1}{4}, \frac{1}{2})$, it is straightforward to prove separately in these two intervals that the right-hand side of (17) is positive. Therefore $g(\alpha)$ as well as $h(\alpha)$ is an increasing function of α in $(0, \frac{1}{2})$. Because of these properties, if we can choose a decreasing sequence $\{\alpha_n\}$ of values for α such that

$$g(\alpha_{n+1}) \geqslant h(\alpha_n), \quad (18)$$

then the inequality (13) holds in the whole interval $[\alpha_{n+1}, \alpha_n]$.

By numerical evaluation the inequality (13) is satisfied for $\alpha = \alpha_0 \equiv \frac{1}{2}$, and (18) is satisfied for $n=0$ if $\alpha_1 = 0.3$. Hence (12) is satisfied in the interval $[0.3, 0.5]$. Similarly, (18) holds for $n=1$, if we choose $\alpha_2 = 0.17$ so that now (12) holds in $[0.17, 0.5]$. Repeating this process we are able to prove (12) for the whole interval $[0.065, 0.5]$. We are now in a position to prove the following theorem.

THEOREM 2. *For* $0 < \alpha < 1$ *and* $\theta_1 \leqslant \theta \leqslant \pi - \theta_1$,

$$B_n(\cos\theta) \geqslant A_n(\cos\theta), \qquad n = 2, 3, 4, \ldots \quad (19)$$

where A_n, B_n *are the bounds to* $P_n^{(\alpha)}(\cos\theta)$ *given by* (1) *and* (3) *respectively.*

Proof.

$$B_n^{-2/\alpha} - A_n^{-2/\alpha} = [P_n^{(\alpha)}(1)]^{-2/\alpha}\left\{1-(1-\cos^2\theta)\right.$$
$$\left.\times\left[\left(\frac{\Gamma(\alpha)(n+\alpha)^{(1-\alpha)}P_n^{(\alpha)}(1)}{2^{1-\alpha}}\right)^{2/\alpha} - \frac{P_n^{(\alpha)}(1)'}{\beta P_n^{(\alpha)}(1)}\right]\right\}, \quad (20)$$

where β is defined in (4). Now $\pi/(n+1) > \theta_1 > \pi/2n$ so that for $\theta_1 \leqslant \theta \leqslant \pi - \theta_1$,

$$(1-\cos^2\theta) \geqslant \sin^2\theta_1 \geqslant \left[\frac{\pi}{2n} - \frac{\pi^3}{48n^3}\right] \geqslant \frac{1.986}{n^2}, \qquad n = 2, 3, 4, \ldots. \quad (21)$$

When $\frac{1}{2}\leqslant\alpha\leqslant 1$ we have defined $\beta=\alpha$ and then using Lemma 1,

$$B_n^{-2/\alpha}-A_n^{-2/\alpha}\leqslant[P_n^{(\alpha)}(1)]^{-2/\alpha}\left\{1-\frac{1.986}{4}\left[\frac{\pi^{1/\alpha}}{(\Gamma(\alpha+\frac{1}{2}))^{2/\alpha}}-\frac{4}{\alpha(2\alpha+1)}\right]\right\}. \tag{22}$$

But from Lemma 3,

$$\begin{aligned} G(\alpha)&\equiv 1+0.4965\left[\frac{4}{\alpha(2\alpha+1)}-\left(\frac{\pi}{\Gamma(\alpha+\frac{1}{2})^2}\right)^{1/\alpha}\right]\\ &\leqslant 1+0.4965\left[-4+\frac{4}{2(2\alpha+1)}\right] \end{aligned} \tag{23}$$

and the R.H.S. of (23) is negative for $0.8\leqslant\alpha\leqslant 1$. Again using Lemma 3 for $\alpha<0.8$,

$$G(\alpha)\leqslant 1+0.4965\left[\frac{4}{\alpha(2\alpha+1)}-\left(\frac{\pi}{\Gamma(1.3)^2}\right)^{1.25}\right] \tag{24}$$

and the R.H.S. of (24) is negative for $0.505\leqslant\alpha<0.8$. Repeating the process once more, we finally prove the R.H.S. of (22) is negative and hence (19) holds for all $\frac{1}{2}\leqslant\alpha\leqslant 1$. When $0<\alpha<\frac{1}{2}$ the R.H.S. of (20) will be negative for $\theta_1\leqslant\theta\leqslant\pi-\theta_1$ if

$$1-\frac{1.986}{n^2}\left\{\left[\frac{\Gamma(n+2\alpha)\,\Gamma(\alpha)(n+\alpha)^{(1-\alpha)}}{\Gamma(n+1)\,2^{(1-\alpha)}\Gamma(2\alpha)}\right]^{2/\alpha}-\frac{n(n+2\alpha)}{\beta(2\alpha+1)}\right\}<0,$$

which will be the case from Lemma 2 if

$$1-\frac{1.986(n+\alpha)^2}{n^2}\left\{\left[\frac{\Gamma(2+2\alpha)\,\Gamma(\alpha)(2+\alpha)^{1-2\alpha}}{\Gamma(3)\,2^{(1-\alpha)}\Gamma(2\alpha)}\right]^{2/\alpha}-\frac{1}{\beta(2\alpha+1)}\right\}<0. \tag{25}$$

$n=2, 3, 4,\ldots$

Using the fact that $(\Gamma(2\alpha)/\Gamma(\alpha))=(2^{(2\alpha-1)}/\pi^{\frac{1}{2}})\,\Gamma(\alpha+\frac{1}{2})$ it follows after some algebra that (25) holds if,

$$\beta\geqslant\frac{1}{2(\alpha+1)}\left\{\left[\frac{\Gamma(\alpha+1)(1+2\alpha)(2+\alpha)^{(1-2\alpha)}}{2^{(1-\alpha)}}\right]^{2/\alpha}-0.503\right\}. \tag{26}$$

This inequality is satisfied for $\beta=\alpha$ when $0.065\leqslant\alpha<\frac{1}{2}$ as follows from Lemma 4. For smaller values of α we take β equal to the R.H.S. of (26) or equal to α depending on which is the larger. In either case (19) is then satisfied.

The results of Theorems 1 and 2 may be combined to give the uniform bound (3) on $P_n^{(\alpha)}(\cos\theta)$ for the whole interval $[0, \pi]$. The only small point to note is that when $0 < \alpha < 0.065$ and $0 \leqslant \theta \leqslant \theta_1$, the bound (5) may be worsened by replacing the factor $1/\alpha$ multiplying $(1 - \cos^2\theta)$ by $1/\beta$ to give the uniform bound (3) since for these values of α, β may be greater than α.

Corresponding bounds on Bessel functions are given by the following simple corollary.

COROLLARY. *For all real* z,

$$|J_{\alpha-\frac{1}{2}}(z)| \leqslant |z/2|^{\alpha-\frac{1}{2}} \Gamma(\alpha+\tfrac{1}{2})^{-1}\{1 + z^2/[(2\alpha+1)\beta(\alpha)]\}^{-\alpha/2} \tag{27}$$

when $0 < \alpha < 1$, *where* $\beta(\alpha)$ *is defined in* (4).

Proof. We use the well known limit [5],

$$(\tfrac{1}{2}z)^{-\nu} J_\nu(z) = \lim_{n\to\infty} n^{-\nu} P_n^{(\nu,\mu)}(\cos z/n), \tag{28}$$

where $P_n^{(\nu,\mu)}(x)$ are Jacobi polynomials. If $\nu = \mu = \alpha - \frac{1}{2}$

$$P_n^{(\nu,\mu)}(x) = \frac{\Gamma(n+\alpha+\frac{1}{2})\,\Gamma(2\alpha)}{\Gamma(\alpha+\frac{1}{2})\,\Gamma(n+2\alpha)} P_n^{(\alpha)}(x). \tag{29}$$

Therefore setting $\nu = \alpha - \frac{1}{2}$ in (28) and using (29) with our bound (3) we obtain inequality (27).

3. CONCLUSIONS

We have generalized the "uniform bounds" obtained by Martin [4] for Legendre polynomials to the case of ultraspherical polynomials. The "nice features" of the former bounds have been preserved except for the small range of values of α close to zero. The problem here is that the bound (5), in the region up to the first zero of $P_n^{(\alpha)}(\cos\theta)$, is then "too strong" to be continued to the whole interval $[0, \pi]$. We have chosen to deal with this difficulty by giving up the requirement that the bound has the same derivative as the polynomial itself at $\theta = 0$, but there may be other ways of handling the problem.

REFERENCES

1. L. LORCH, Inequalities for ultraspherical polynomials and the gamma function, *J. Approx. Theory* **40** (1984), 115–120.

2. G. SZEGÖ, "Orthogonal Polynomials", Amer. Math. Soc., Colloq. Publ. Vol. 23, 4th ed., Providence, R.I., 1975.
3. V. A. ANTONOV AND K. V. HOLŠENIKOV, An estimate of the remainder in the expansion of the generating function for the Legendre polynomials (Generalisation and Improvement of Bernstein's inequality), *Vestnik Leningrad Univ. Math.* [English trans.] **13** (1981), 163–166.
4. A. MARTIN, Unitarity and high energy behaviour of scattering amplitudes, *Phys. Rev.* **129** (1963) 1432–1436.
5. W. MAGNUS, F. OBERHETTINGER, AND R. P. SONI, "Formula and Theorems for the Special Functions of Mathematical Physics," Springer-Verlag, Berlin, 1966.